Appendix B: Notes on Linear Viscoelasticity

Supplementary material for the course of Solid Mechanics

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APPENDIX B: Notes on linear viscoelasticity

B.1. Introduction

In a typical problem of solid mechanics, we are interested in calculations of the displacement, strain and stress fields (as a function of time) of a body. These are: 3 components of displacements $u_i(x_i)$, 6 components of strains $\varepsilon_{ij}(x_i)$, 6 components of stresses $\sigma_{ij}(x_i)$. These are 15 unknowns to be determined. On the other hand, the available equations are 6 strain displacement relations, and 3 equations of equilibrium. Thus, we have 15 unknowns and 9 equations. As such the problem is not possible to solve.

What we need to supply to this system of 9 equations is a set of 6 equations that relate the stresses and strains. These so-called constitutive equations (stress-strain, or strain-stress relations) depend on the material under study and include a number of material parameters. Since these equations cannot be derived on the basis of first principles, we resort to laboratory experiments where the constitutive response is measured under simple load cases and, for the same material, is generalized for complicated load cases. With these relations established, the system of equations is now complete and one seeks its solution for the 15 unknowns by accounting for the boundary conditions of the problem at hand.

In solid mechanics, one of the simplest cases of stress - strain relation is when there is a linear dependence of stresses on strains or visa-versa. Such linear relations are found in most materials under relatively low levels of loads or displacements. The underlying theory is the so called theory of linear elasticity and is used to obtain satisfactory solutions in various problems. Due to its importance in engineering applications, this theory is treated with some details on a separate document.

The stress-strain behavior of many materials is in general nonlinear, inelastic and anisotropic. A few examples are shown in Fig. B1. As a result, several models have been developed and presently used to analyze certain aspects of material response. Among these approaches, we have linear and non-linear elasticity, viscoelasticity, plasticity and viscoplasticity. Each of them aims at approximately modeling some specific aspects of the actual material behavior and used to obtain engineering answers to a certain class of problems. In this Appendix, we discuss some basic elements of linear viscoelasticity. With this theory we aim at modeling the time-dependent constitutive response of material.

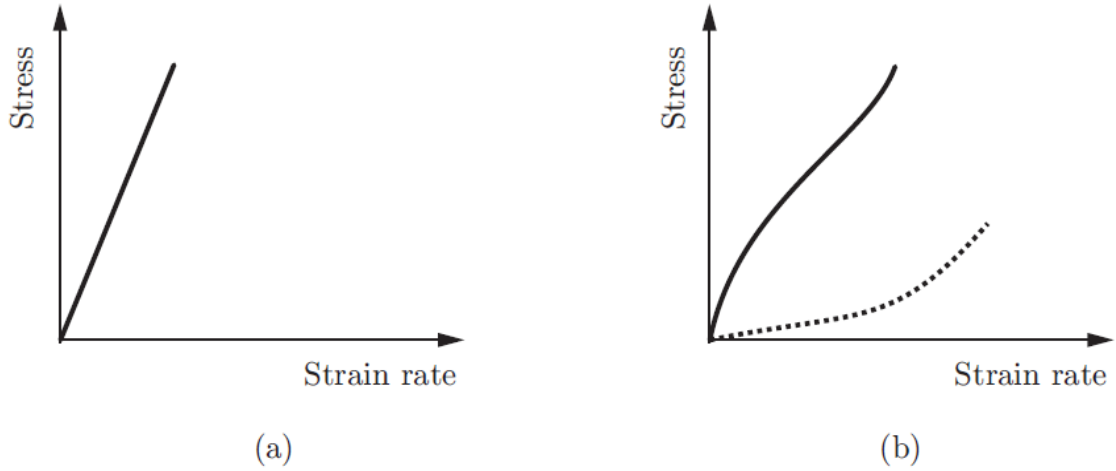

Fig. B1a: Stress-strain rate relations, (a) linear, (b) non-linear (Botsis and Deville).

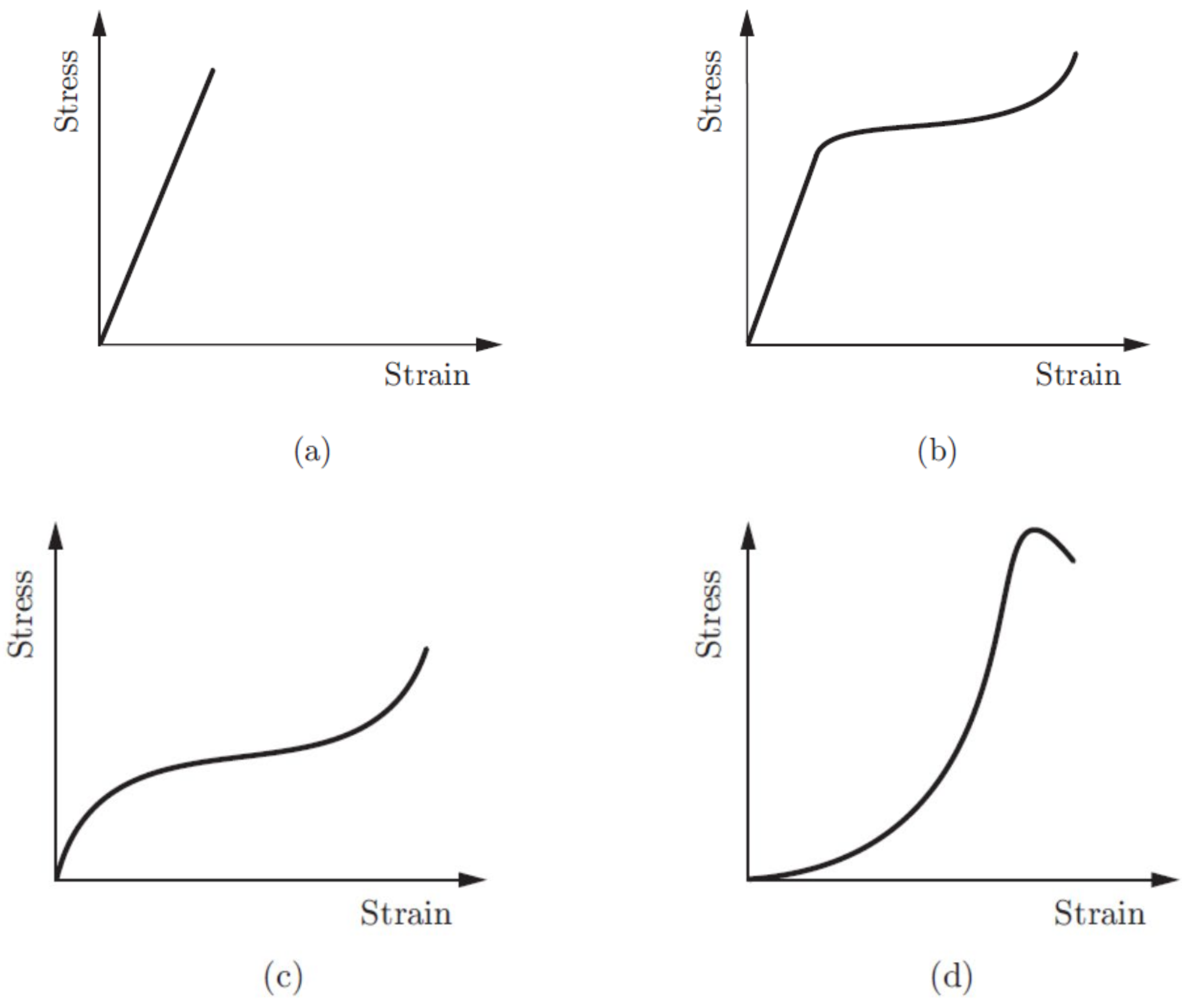

Fig. B1b: various types of stress-strain response. (a) solid linear elastic, (b) elastoplastic, (c) elastomer, (d) a biological tissue (ref. 4).

B.2. Basic elements of linear visoelasticity

A material is said to be *viscoelastic* if its dominant mechanical behavior presents both *elastic* and *viscous* characteristics. Several engineering materials, such as rubbers, gels, polymer and composite materials, concrete, metals at elevated temperatures, exhibit time dependent response. With the theory of viscoelasticity, we model the response of material that exhibit elastic and viscous characteristics. In polymeric materials, such behavior is associated with movements of polymer molecules. In other materials, the mechanisms are different and are not necessarily related to viscosity. In viscoelasticity, the characteristics of time-dependent material response are conventionally identified by *creep*, *creep-recovery* and *relaxation* experiments.

The objective here is to present some basic notions from the classical theory of viscoelasticity. In what follows, we suppose that deformations are *infinitesimal*, i.e.,

$$\left\| \frac{\partial u_i}{\partial x_j} \right\| = O(\varepsilon) << 1; \quad F_{ij} = \delta_{ij} + O(\varepsilon); \quad J \approx 1 + O(\varepsilon); \quad (i, j = 1,2,3).$$

Here F_{ij} are the matrix components of the deformation gradient tensor and J is its Jacobian. In addition, we shall be limited to *linear* viscoelastic phenomena. One of the consequences of the infinitesimal deformation hypothesis is that the principle of objectivity is no longer needed to be taken into account.

A linear elastic response, described by Hook's law, describes a time independent linear relationship between the stresses and strains. Furthermore, it implies that the response to a given input is instantaneously realized, or there is no phase lag between input and output. A linear viscoelastic response also suggests a linearity between stresses and strains. This relationship, however, is a functional of the load-time history.

The basic criteria in the linear theory of viscoelasticity are two: that of proportionality and that of superposition. To state these two criteria let us indicate with I the input and the corresponding response with R and their relationship as, $R = R[I]$ where $R[I]$ indicates the current value of R as a functional of the time history of input I. If the solid is viscoelastic, the response $R[I]$ must satisfy the following two conditions,

- Proportionality: $R[cI] = cR[I]$ (where c is a constant).
- Superposition: $R[I_a + I_b] = R[I_a] + R[I_b]$,

where I_a, I_b are different time histories. Based on two criteria we can define the stress-strain relationship, which is well known as the Boltzmann superposition integral, that we define next. Consider the dimensional input stress shown in Fig. B2a under isothermal condition.

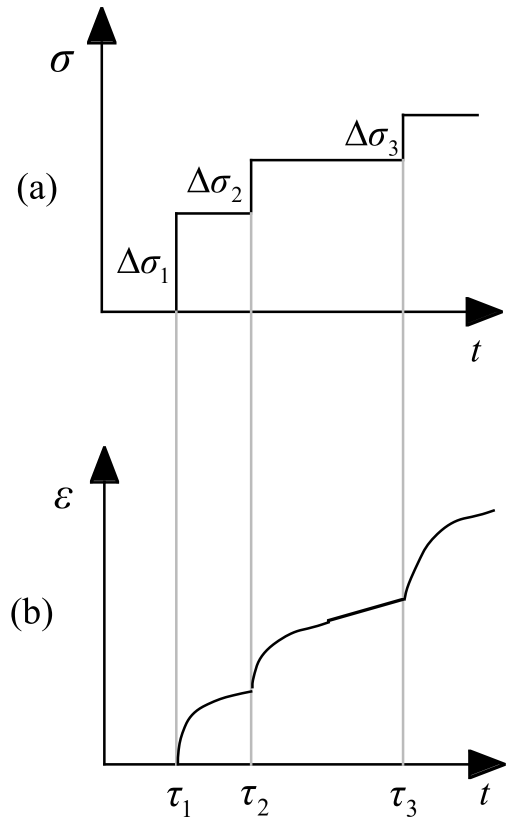

Fig. B2: Input stress (a) and response strain (b).

According to Boltzmann's superposition principle, the response, i.e., strain, is linear and proportional to the input stress, the proportionality factor being a function of the elapsed time since the input stress. Accordingly, for the input history shown in Fig. B2a, the total strain response at any time $t > \tau_3$ is,

$$\varepsilon(t) = \Delta\sigma_1 J(t-\tau_1) + \Delta\sigma_2 J(t-\tau_2) + \Delta\sigma_3 J(t-\tau_3) \tag{B.1}$$

where $J(t)$ is the *creep compliance,* or *creep function*. When the input stresses have arbitrary time histories, (B.1) is given by the Boltzmann superposition integral,

$$\varepsilon(t) = \int_{-\infty}^{t} J(t-\tau)\frac{d\sigma(\tau)}{d\tau}d\tau . \tag{B.2a}$$

Similarly, the stress that results from an arbitrary strain input is given by,

$$\sigma(t) = \int_{-\infty}^{t} E(t-\tau)\frac{d\varepsilon(\tau)}{d\tau}d\tau \tag{B.2b}$$

where $E(t)$ is the *relaxation modulus, or relaxation function*. Thus, when the stress (or strain) history in $(-\infty, t]$ is know the strain (or stress) history can be determined with these integrals.

The two Boltzmann superposition integrals can be transformed to ordinary differential equation having the derivatives of stress and strain in terms of time. The differential equations make it easier to construct models and understand viscoelastic behavior better. To convert the Boltzmann superposition integrals to an ordinary differential equation we use the techniques of Laplace transform.

The Laplace transform of a function $f(t)$, designated as $L\{f(t)\}$ or $\overline{f}(s)$, is defined as,

$$L\{f(t)\} = \bar{f}(s) = \int_0^\infty f(t)e^{-st}ds \tag{B.3}$$

where *s* is the Laplace parameter which can be real or complex number.

In mathematical terms, with the Laplace transform, we "transform" the equations to another space where it is more convenient to solve it. Once we have the solution we can transform them back to the original space using the inverse transformation. In the particular case we go from the time domain t, to a domain s. We proceed now to examine the Laplace transform of the superposition integral given by (B.2b),

$$L\{\sigma(t)\} = \overline{\sigma}(s) = L\left\{\int_{-\infty}^{t} E(t-\tau)\frac{d\varepsilon(\tau)}{d\tau}d\tau\right\}. \tag{B.4}$$

The right hand side of equation (B.4) is in the form of a convolution integral. Thus, we can write,

$$\overline{E}(s)\frac{d\overline{\varepsilon}(s)}{d\tau} = L\left\{\int_{-\infty}^{t} E(t-\tau)\frac{d\varepsilon(\tau)}{d\tau}d\tau\right\}. \tag{B.5}$$

By taking the inverse transform of (B.5), we can write,

$$L^{-1}\left\{\overline{E}(s)\frac{d\overline{\varepsilon}(s)}{d\tau}\right\} = \int_{-\infty}^{t} E(t-\tau)\frac{d\varepsilon(\tau)}{d\tau}d\tau\ . \tag{B.6}$$

Accounting for (B.6), equation (B.4) can be written as follows,

$$\overline{\sigma}(s) = L\left[L^{-1}\left\{\overline{E}(s)\frac{d\overline{\varepsilon}(s)}{d\tau}\right\}\right] = \overline{E}(s)\frac{d\overline{\varepsilon}(s)}{d\tau}. \tag{B.7}$$

From the properties of Laplace transform for the derivatives, we can write,

$$L\left\{\frac{d\varepsilon(\tau)}{d\tau}\right\}=\frac{d\bar{\varepsilon}(s)}{d\tau}=s\bar{\varepsilon}(s)-\varepsilon(0) \tag{B.8}$$

where $\varepsilon(0)$ is the initial strain (at t = 0). Considering $\varepsilon(0)=0$ and combining the last two equations, we have,

$$\bar{\sigma}(s)=s\bar{E}(s)\bar{\varepsilon}(s). \tag{B.9}$$

Following the same procedure with (B.2a), we obtain,

$$\bar{\varepsilon}(s)=s\bar{J}(s)\bar{\sigma}(s). \tag{B.10}$$

Interestingly, equations (B.9) and (B.10) resemble Hook's law in that there is a linearity between the stress and strain. The proportionality constants being the Laplace transform of the creep and relaxation moduli. This observation indicates a correspondence between the equations of linear elasticity and linear viscoelasticity. To proceed further, we recognize that equation (B.10) can be written as,

$$\bar{\varepsilon}(s)=s\bar{J}(s)\bar{\sigma}(s)=\frac{Q(s)}{P(s)}\bar{\sigma}(s) \tag{B.11a}$$

where,

$$Q(s)=b_o+b_1s+b_2s^2+\ldots+b_ns^n; \qquad P(s)=a_o+a_1s+a_2s^2+\ldots+a_ns^n..$$

and $b_o,b_1,\ldots,b_n,\ a_o,a_1,\ldots,a_n$ are constants related to material.

Equation (B.11a) can be written as,

$$P(s)\bar{\varepsilon}(s)=Q(s)\bar{\sigma}(s). \tag{B.11b}$$

From the properties of Laplace transform and neglecting the initial conditions we can write for a function $f(t)$,

$$L\left\{\frac{d^n f(t)}{dt^n}\right\}=s^n\bar{f}(s). \tag{B.12}$$

Taking the Laplace transform of (B.11b) and using (B.12) we can write,

$$
\begin{aligned}
&b_o\sigma(t)+b_1\frac{d\sigma(t)}{dt}+b_2\frac{d^2\sigma(t)}{dt^2}+...+b_n\frac{d^n\sigma(t)}{dt^n} \\
&=a_o\varepsilon(t)+a_1\frac{d\varepsilon(t)}{dt}+a_2\frac{d^2\varepsilon(t)}{dt^2}+...+a_n\frac{d^2\varepsilon(t)}{dt^2}
\end{aligned}
\tag{B.13}
$$

The last equation demonstrates that a linear viscoelastic response can be described either by the Boltzmann's superposition integrals (B.2) or a differential equation (B.13). For example, in linear elastic behavior we have time independent response $a_o\varepsilon = b_o\sigma$, which is a special case of (B.13) when all time derivatives are zero.

B.3. Viscoelastic models

In viscoelasticity we combine two mechanical analogues to develop constitutive models. These are, the spring portraying a linear elastic response, and the dashpot that represents a viscous/fluid response.

More precisely, for a linear spring (Fig. B3), the stress is related to strain by Hooke's law

$$\sigma_e = E\varepsilon_e \tag{B.14}$$

where σ_e denotes the Cauchy stress applied to the spring, ε_e the resulting elastic strain, and E the elastic modulus. For the dashpot (Fig. B3), the stress is linearly related to the rate of strain,

$$\sigma_v = \eta\frac{d\varepsilon_v}{dt}, \quad \Rightarrow \varepsilon_v = \frac{\sigma_v}{\eta}t \tag{B.15}$$

where σ_v is the Cauchy stress applied to the dashpot, ε_v the viscous strain and η the coefficient of viscosity. These two elements have distinct response to mechanical stimuli:

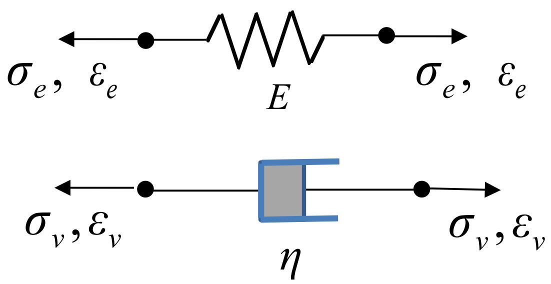

Fig. B3: Spring (a) and Dashpot (b).

- Upon application of mechanical stimulus, the spring exhibit instantaneous elasticity and instantaneous recovery upon removal of the stimulus.

- According to (B.15), the dashpot deforms continually upon stress at a constant rate when is subjected to stress. However, it is impossible to apply a finite instantaneous stimuli to it and thus, stress relaxation (i.e. sudden application of strain) is not realistic on a dashpot.

Depending on the experimental conditions, combination of these two elements can give realistic constitutive models for the materials' time dependent response. Amongst the simplest ones are the known in the literature as *Maxwell* and *Kelvin-Voigt* models. Below, we present them and introduce the important experiments, *creep* and *relaxation*.

Maxwell Model. This model is composed of a spring and dashpot *in series* (Fig. B4). In this model, the stress σ_e in the spring and the stress σ_v in the dashpot are equal to the applied stress σ,

$$\sigma = \sigma_e = \sigma_v \tag{B.16}$$

However, the total strain ε is the sum of the spring strain and the dashpot strain,

$$\varepsilon = \varepsilon_e + \varepsilon_v \tag{B.17}$$

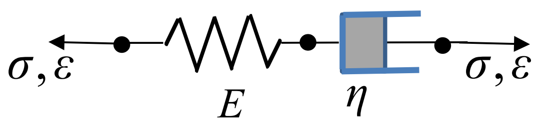

Fig. B4: The Maxwell model.

Combining equations (B.14) to (B.17), we obtain the constitutive equation for the Maxwell model,

$$\dot{\varepsilon} = \frac{d\varepsilon}{dt} = \frac{d\varepsilon_e}{dt} + \frac{d\varepsilon_v}{dt} = \frac{1}{E}\frac{d\sigma}{dt} + \frac{\sigma}{\eta} = \frac{\dot{\sigma}}{E} + \frac{\sigma}{\eta}$$

or,

$$\dot{\varepsilon} = \frac{\dot{\sigma}}{E} + \frac{\sigma}{\eta}. \tag{B.18}$$

We see that this equation is a special case of equation (B.13).

To simply writing the differential equations, the time derivatives of strains and stress are shown in the following as $\dot{\varepsilon}$ and $\dot{\sigma}$.

Kelvin-Voigt model. It consists of a spring and dashpot *in parallel* (Fig. B5).

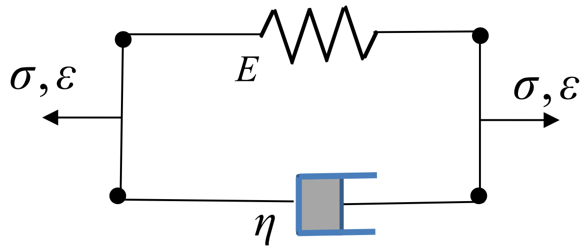

Fig. B5: The Kelvin-Voigt model.

Because of this parallel arrangement, the spring strain ε_e and the dashpot strain ε_v are the same as the total strain while the total stress is the sum of the corresponding stresses at each element,

$$\varepsilon = \varepsilon_e = \varepsilon_v \qquad \text{(B.19)}$$

$$\sigma = \sigma_e + \sigma_v \,. \qquad \text{(B.20)}$$

These two equations together with equations (B.14) and (B.15) allow us to obtain the constitutive equation for the Kelvin-Voigt model,

$$\sigma = E\varepsilon + \eta\dot{\varepsilon} \qquad \text{(B.21)}$$

This is another particular case of equation (B.13).

B.4. Experimental observations

The time-dependent material response is characterized by *creep*, *relaxation* and *creep-recovery* experiments, carried out in a controlled laboratory environment. Since the temperature has an essential role in the response, it is normally controlled and is part of the recorded data. Dynamic testing is also very important but it will not be addressed in this short discussion.

In a creep, the tested specimen is subjected to stress σ_o which is maintained *constant* during the experiment. Simultaneously, the time-dependent strain $\varepsilon(t)$ is recorded.

Typical experimental data for High Density Polyethylene (HDPE) under different stress levels σ_0 are shown in Fig. B6. The resulting plot of creep strain $\varepsilon(t)$ versus t is called a *creep curve*. When the material is linearly viscoelastic, $\varepsilon(t)$ is in linearly proportional to σ_0. In such a case, the function $J(t)$, defined by,

$$J(t) = \varepsilon(t) / \sigma_0 \tag{B.22}$$

is a characteristic of the material and referred to as the *creep function*. Often, a creep modulus is also defined as the inverse of the creep function.

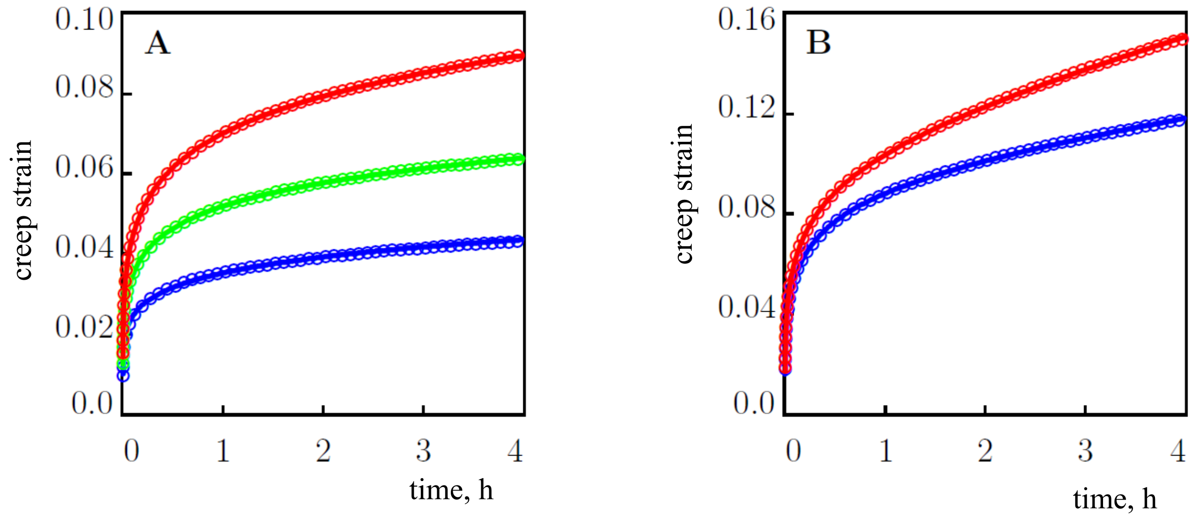

Fig. B6: Creep of High Density Polyethylene at 21°C under different stresses (A): 9.0 (blue), 11 (green), 13 (red); (B) 14. (blue), 15 (red). (units in MPa, From ref. 1).

Experimental data of stress relaxation for two materials are shown in Fig. B7 and Fig. B8. When a relaxation experiment is carried out, the tested specimen is subjected to a constant strain ε_o and resulting graph $\sigma = \sigma(t)$ vs time is named *relaxation curve*. When the material is linearly viscoelastic, the function defined as,

$$G(t) = \sigma(t) / \varepsilon_o \tag{B.23}$$

is called the *relaxation function*. It can be shown that for a given material, its creep and relaxation functions $J(t)$ and $G(t)$ are related (Christensen, 1971).

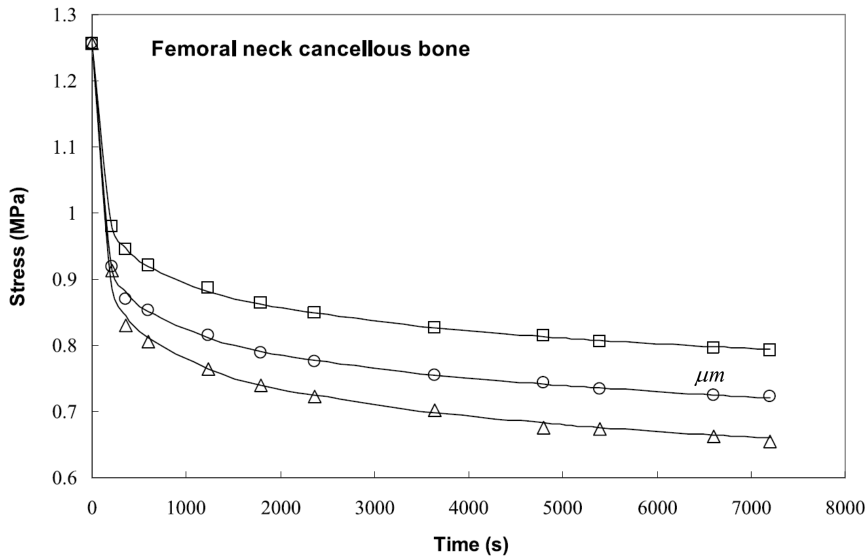

Fig. B7: Stress relaxation of bone at 21°C under three initial strains (from ref. 3).

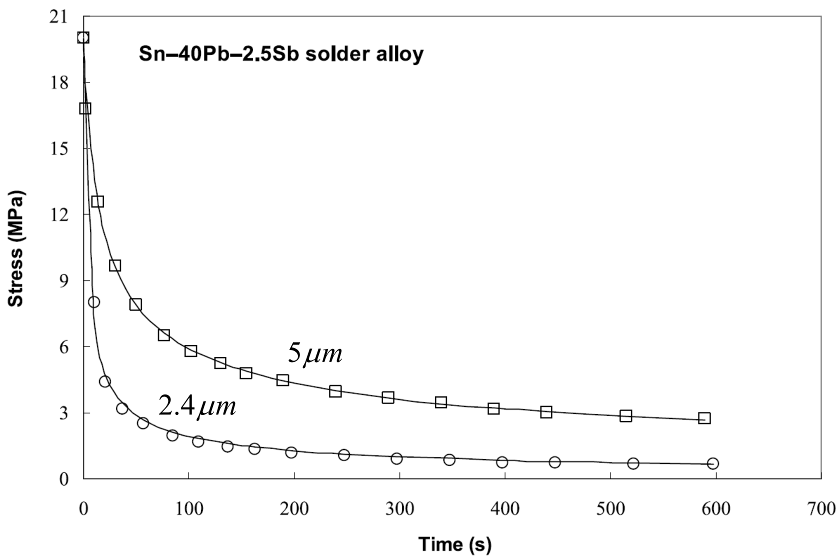

Fig. B8: Stress relaxation solder alloy at 21°C with two different grain size (from ref. 4).

The question that arises next is: what are the models to describe these experimental (creep and relaxation) data shown in the preceding figures? It was mentioned earlier that springs and dashpots are combined to construct pertinent models. Such combinations and experimental evidence can lead to satisfactory viscoelastic models. In the following, we discuss how to build such models followed by their application to creep and/or relaxation. However, it is not the purpose to analyze specific experimental data.

Creep and relaxation with Maxwell model.

(a) Creep response. For this, a stress σ_o is applied at time $t = 0$ and maintained constant. In this case, equation (B.18) has the solution,

$$\varepsilon(t) = \frac{\sigma_o}{\eta} t + c \tag{B.24}$$

where c is a constant of integration. This constant is determined by the initial condition:

$$\varepsilon_0 = \varepsilon(0) = \frac{\sigma_o}{E} \tag{B.25}$$

where ε_0 corresponds to the instantaneous elastic strain of the spring under the action of σ_o.

Introducing (B.25) into (B.24) gives,

$$c = \frac{\sigma_o}{E}. \tag{B.26}$$

Thus, the creep response of the Maxwell model is (Fig. B9a),

$$\varepsilon(t) = \left(\frac{t}{\eta} + \frac{1}{E} \right) \sigma_o \tag{B.27}$$

with the associated creep function given by,

$$J(t) = \frac{\varepsilon(t)}{\sigma_o} = \frac{t}{\eta} + \frac{1}{E} \tag{B.28}$$

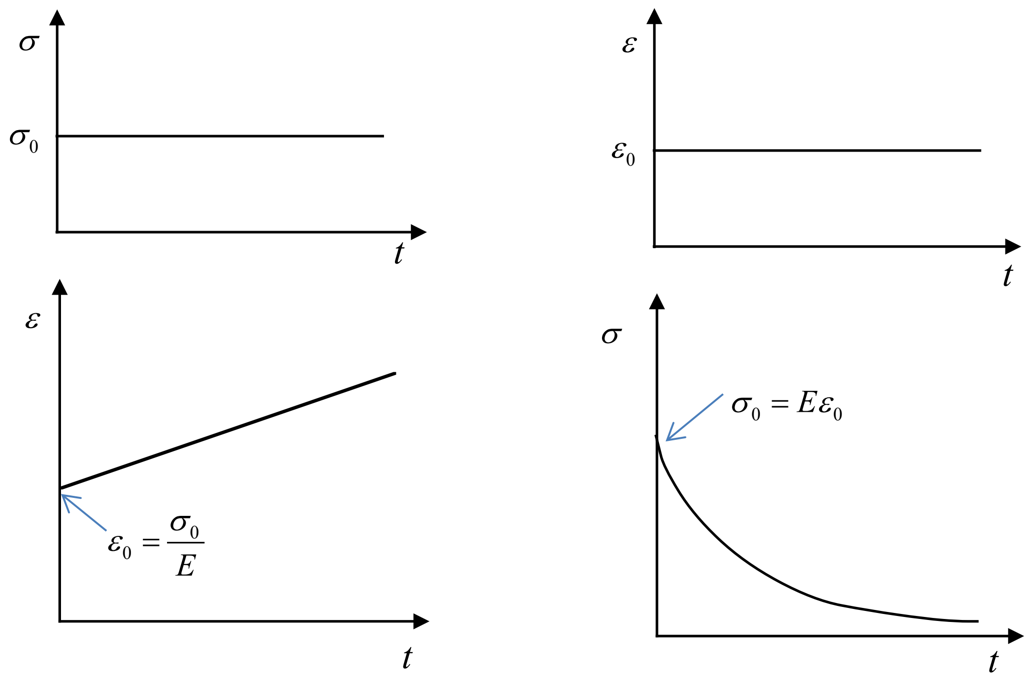

Fig. B9: Response of the Maxwell model in Creep (a), and Relaxation (b).

(b) Relaxation. If a constant strain ε_o is applied for $t \geq 0$, equation (B.18) reduces to,

$$\dot{\sigma} + \frac{E}{\eta}\sigma = 0 \tag{B.29}$$

With the initial condition,

$$\sigma_o = \sigma(0) = E\varepsilon_o, \tag{B.30}$$

(B.29) has the solution,

$$\sigma(t) = \sigma_o \exp(-tE/\eta) \tag{B.31}$$

which is the relaxation response of the Maxwell model (Fig. B9b). The derivative of the last equation at $t = 0$ gives,

$$\left.\frac{d\sigma}{dt}\right|_{t=0} = -\frac{\sigma_o E}{\eta}.$$

This is the slope of a straight line tangent to $(0, \sigma_o)$. This line intersects the time axis at $t = \tau = \eta / E$ where τ is the so-called *relaxation time*. Note that when $\tau >> t$, the elastic response dominates while $\tau << t$, the viscous one is dominant. When $\tau \approx t$, both elasticity and viscous are shown. For $\tau = t$, the stress is given by (B.31) $\sigma(\tau) = \sigma_o \exp(-\tau/\tau) = \sigma_o / e$. Note that the corresponding relaxation function is obtained by,

$$G(t) = \frac{\sigma(t)}{\varepsilon_o} = Ee^{-tE/\eta} \tag{B.32}$$

Creep and relaxation with Kelvin-Voight model.

(a) Creep. Let us study now the creep response of Kelvin-Voigh model (Fig. B5). If $\sigma = \sigma_o$ for $t > 0$, (B.21) becomes,

$$\eta\dot{\varepsilon} + E\varepsilon = \sigma_o \tag{B.33}$$

whose solution is,

$$\varepsilon(t) = ce^{-tE/\eta} + \frac{\sigma_o}{E} \tag{B.34a}$$

with the initial conditions $\varepsilon_0 = \varepsilon(0) = 0$, it becomes,

$$\varepsilon(t) = \frac{\sigma_o}{E}\left[1 - e^{-t/\tau}\right] \tag{B.34b}$$

with $\tau = \eta / E$. Then the creep function for the Kelvin-Voigt model is given by,

$$J(t) = \frac{\varepsilon(t)}{\sigma_o} = \frac{1}{E}\left[1 - e^{-t/\tau}\right] = J\left[1 - e^{-t/\tau}\right] \tag{B.35}$$

where a compliance parameter $J = 1/E$ is also used for consistency in writing the equations. The function $\varepsilon(t)$ is illustrated in Fig. B10. As $t \to \infty$, the strain approaches a constant value $\varepsilon_\infty = \varepsilon_0$ which is proportional to σ_o,

$$\varepsilon_\infty = \varepsilon_0 = \lim_{t\to\infty} \varepsilon(t) = \frac{\sigma_0}{E}. \tag{B.36}$$

Such response behavior is said to be *delayed elastic*. As in the case of the relaxation time, defined earlier in the relaxation of Maxwell model, a similar interpretation of the characteristic time $\tau = \eta / E$ is given and called here the *retardation time*. The derivative of (B.34b) at the origin gives the slop of a straight line with slope $d\varepsilon / dt\big|_{t=0} = \sigma_0 / E\tau$. It is not difficult to see that this line intersects line $\varepsilon(t) = \varepsilon_\infty$ when $t = \tau$. For $t \ll \tau$ the viscous component is important, while for $t \gg \tau$ the elasticity dominates and when $t \approx \tau$ both mechanisms are shown and for $t = \tau$,

$$\varepsilon(\tau) = \frac{\sigma_o}{E}(1 - 1/e) = 0.63\frac{\sigma_o}{E}$$

(b) Relaxation. The relaxation response of the Kelvin-Voigt model cannot be treated in the same manner as that used for the Maxwell model. Indeed, due to the parallel arrangement of the spring and the dashpot, the spring can only extend slowly so that no instantaneous strain can be produced. Therefore, we cannot impose a constant step-strain ε_0 for $t > 0$. However, it is possible to progressively stretch the spring for the strain to attain the value in certain experiments.

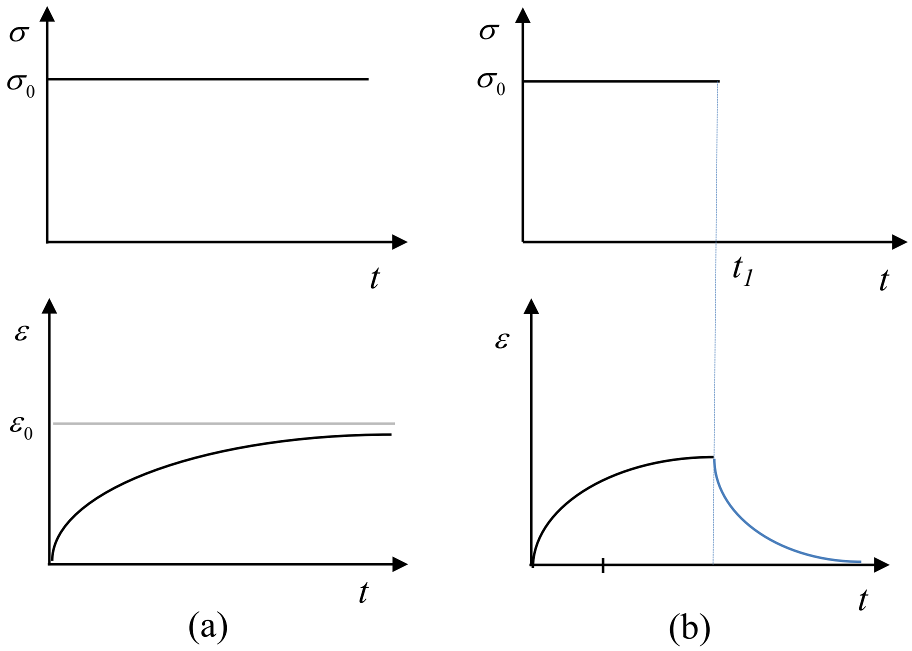

Fig. B10: Creep (a) and Creep-recovery (b) of Kelvin-Voight Model.

Creep recovery with Kelvin-Voight model.

Another important experiment in viscoelasticity is the *creep-recovery* experiment. Firstly, a creep experiment is performed for a certain time. Secondly, the stress is removed and the material is allowed to recover its initial shape. Here we illustrate this experiment using Kelvin-Voight model.

Upon application of σ_0 on the Kelvin-Voight model in phase 1, the strain is given by,

$$\varepsilon_1(t) = \frac{\sigma_0}{E}\left[1 - e^{-tE/\eta}\right]. \qquad \text{(B.34bis)}$$

At a certain time t_1 the stress is removed. The resulting strains are given by the same equation for an applied stress $-\sigma_0$, i.e.,

$$\varepsilon_2(t) = -\frac{\sigma_o}{E}\left[1 - e^{-(t-t_1)E/\eta}\right], \qquad t > t_1.$$

According to the principle of superposition, the strain during the recovery phase is given by,

$$\varepsilon_1(t) + \varepsilon_2(t) = \frac{\sigma_o}{E} e^{-tE/\eta}\left[e^{+t_1 E/\eta} - 1\right], \qquad t > t_1$$

with the property that $t \to \infty, \quad \varepsilon_\infty \to 0$. The behavior is schematically drawn in Fig. B10b.

If we compare the experimental data in Figs B6 and B7, with what the two simple models predict, it is clear that the simple Maxwell and Kelvin-Voight models (Figs B9 and B10) are not adequate to describe time-dependent response of real materials. That is why models in various combinations with springs and dashpots have been proposed to examine specific time-dependent constitutive response. In what follows, we present as examples some of the well-known models.

B.5. Examples

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1. Constitutive equation for a three-parameter model, Maxwell presentation.

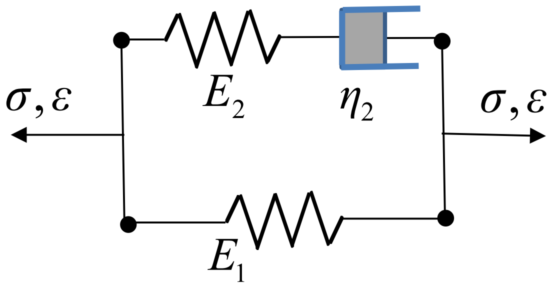

Fig. B11: Maxwell model and a spring in parallel.

The configuration shown in Fig. B11 consists of a spring and a Maxwell model in parallel. Due to the parallel setting, the spring and Maxwell's model are subjected to the same strain ε and with the stresses expresses as,

$$\sigma = \sigma_1 + \sigma_2 . \quad \text{(a)}$$

For the Maxwell model,

$$\dot{\varepsilon} = \frac{\dot{\sigma}_2}{E_2} + \frac{\sigma_2}{\eta_2} . \quad \text{(b)}$$

For the spring,

$$\varepsilon = \frac{\sigma_1}{E_1} \quad \text{(c)}$$

From (a) we have,

$$\sigma_2 = \sigma - \sigma_1 . \quad \text{(d)}$$

Combining (c) and (d) we obtain,

$$\sigma_2 = \sigma - \sigma_1 = \sigma - \varepsilon E_1 \quad \Rightarrow \quad \dot{\sigma}_2 = \dot{\sigma} - \dot{\varepsilon} E_1 \tag{e}$$

Introducing (e) in (b) we have,

$$\dot{\varepsilon} = \frac{\dot{\sigma} - \dot{\varepsilon} E_1}{E_2} + \frac{\sigma - \varepsilon E_1}{\eta_2} \quad \Rightarrow \quad \dot{\varepsilon} = \frac{\eta_2 \left(\dot{\sigma} - \dot{\varepsilon} E_1 \right)}{\eta_2 E_2} + \frac{E_2 \left(\sigma - \varepsilon E_1 \right)}{\eta_2 E_2} \tag{f}$$

Rearranging the last relation, we obtain the final equation, conventionally called the equation of state of the *standard solid*, Maxwell representation, i.e.,

$$\sigma + \frac{\eta_2}{E_2} \dot{\sigma} = \frac{\eta_2}{E_2} \left(E_1 + E_2 \right) \dot{\varepsilon} + \varepsilon E_1 . \tag{g}$$

Note that (g) is a particular case of (B.13).

We derive now the relaxation and creep functions using this model.

- **relaxation function**.

With $\dot{\varepsilon} = 0$ and $\varepsilon = \varepsilon_0$, equation (g) becomes,

$$\dot{\sigma} + \frac{E_2}{\eta_2} \sigma - \varepsilon_0 \frac{E_1 E_2}{\eta_2} = 0 .$$

With the definition $\tau_2 = \eta_2 / E_2$, the solution of this ordinary differential equation is,

$$\sigma(t) = e^{-\int dt/\tau_2} \left[c + \varepsilon_0 \frac{E_1 E_2}{\eta_2} \int e^{\int dt/\tau_2} dt \right]$$

$$\sigma(t) = e^{-t/\tau_2} \left[c + \varepsilon_0 E_1 e^{t/\tau_2} \right].$$

For the model in Fig. B11, under relaxation at $t = 0$, we have,

$$\sigma(0) = \varepsilon_0 (E_1 + E_2) \quad \Rightarrow \quad c = \varepsilon_0 (E_1 + E_2) - \varepsilon_0 E_1 = \varepsilon_0 E_2 .$$

Substituting c in the last equation, we have as relaxation function,

$$G(t) = \sigma(t) / \varepsilon_0 = E_1 + E_2 e^{-t/\tau_2} .$$

A schematic of this $G(t)$ vs t is shown in Fig. B12a.

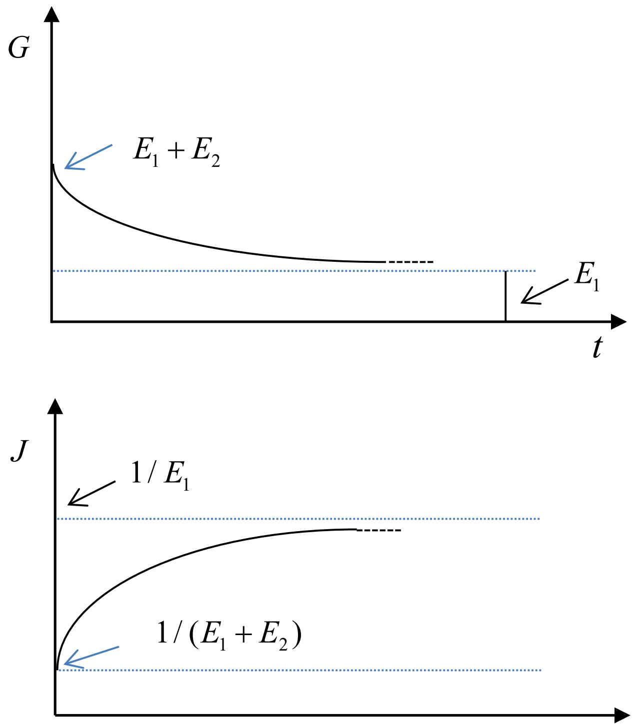

Fig. B12: Relaxation (a) and Creep (b) of the standard model (Fig. B11).

- **creep function**

With $\dot{\sigma} = 0$ and $\sigma = \sigma_0$, equation (g) takes the following form,

$$\sigma_0 = \frac{\eta_2}{E_2}(E_1 + E_2)\dot{\varepsilon} + \varepsilon E_1 \quad \Rightarrow \dot{\varepsilon} + \varepsilon \frac{E_1 E_2}{\eta_2 (E_1 + E_2)} - \sigma_0 \frac{E_2}{\eta_2 (E_1 + E_2)} = 0\,.$$

With $1/\tau_2 = \dfrac{E_1 E_2}{\eta_2 (E_1 + E_2)}$ it becomes,

$$\dot{\varepsilon} + \varepsilon / \tau_2 - \sigma_0 / \tau_2 E_1 = 0$$

which has as a solution,

$$\varepsilon(t) = e^{-t/\tau_2}\left[c + \frac{\sigma_0}{\tau_2 E_1}\int e^{\int dt/\tau_2} dt\right] \quad \Rightarrow \quad \varepsilon(t) = e^{-t/\tau_2}\left[c + \frac{\sigma_0}{E_1} e^{t/\tau_2}\right].$$

For this loading, the model's initial condition, ($t = 0$) is,

$$\varepsilon(0) = \sigma_0 / (E_1 + E_2) \quad \Rightarrow \quad c = \sigma_0 \left[\frac{1}{E_1 + E_2} - \frac{1}{E_1}\right].$$

Thus, the creep function for the model in Fig. B11 is

$$J(t) = \varepsilon(t)/\sigma_0 = \frac{1}{E_1} - \frac{E_2}{E_1(E_1+E_2)} e^{-t/\tau_2}.$$

A schematic of $J(t)$ vs t is shown in Fig. B12b.

Note that the response of the standard model is more realistic than the Maxwell or Kelvin-Voigt models alone when we compare its trend with the experimental data especially in relaxation.

2. Maxwell models in parallel

We saw earlier that the three parameter Maxwell model, describes the trend of a relaxation experiment. In real materials, however (i.e., polymers), several relaxation times are present, related to characteristic times of molecular motion. To capture such behaviors, several Maxwell models and a single spring are arranged in parallel. Fig. B13 shows several such elements and a single spring.

The equation describing each Maxwell model is,

$$\dot{\varepsilon} = \frac{\dot{\sigma}_i}{E_i} + \frac{\sigma_i}{\eta_i} \qquad (i = 1, \ldots, n). \tag{B.18bis}$$

For stress relaxation with $\dot{\varepsilon} = \dot{\varepsilon}_i = 0$ the last equation becomes,

$$\frac{\dot{\sigma}_i}{E_i} + \frac{\sigma_i}{\eta_i} = 0 \quad \Rightarrow \quad \frac{d\sigma_i}{dt} + \frac{E_i}{\eta_i}\sigma_i = 0 \tag{a}$$

or,

$$\ln \sigma_i = -\frac{E_i}{\eta_i} t + c_i. \tag{b}$$

At $t = 0$, we have,

$$\sigma_i = \sigma_{i0} \Rightarrow \ln \sigma_{i0} = c_i \ . \tag{c}$$

Using (c) in (b) and setting $\tau_i = \eta_i / E_i$ we get,

$$\sigma_i(t) = \sigma_{i0} e^{-t/\tau_i} \tag{d}$$

The spring is also subjected to the same strain and thus, the stress is,

$$\sigma_s = E_s \varepsilon_0 . \tag{e}$$

For the entire set of elements, the total stress is the sum of the stresses in (d) and (e),

$$\sigma(t) = \sigma_s + \sum_1^n \sigma_i(t) = \sigma_s + \sum_1^n \sigma_{i0} e^{-t/\tau_i} . \tag{f}$$

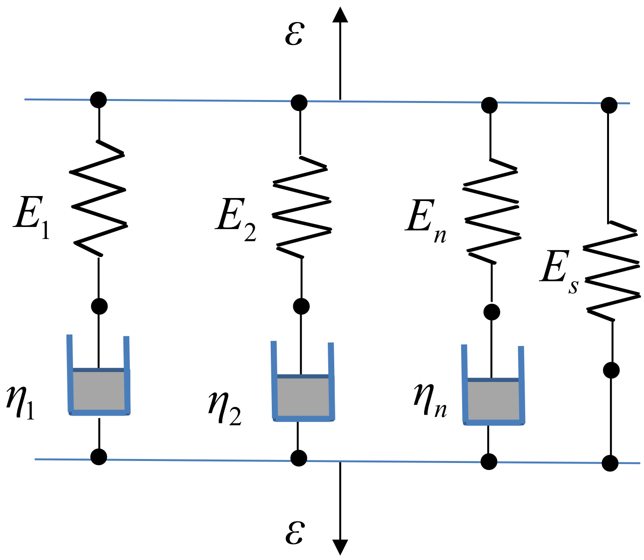

Fig. B13: Maxwell models and spring arranged in parallel.

The stress σ_{i0} at $t = 0$, can be expressed as $\sigma_{i0} = \varepsilon_0 E_i$. Combining, (e) and (f) we obtain,

$$\sigma(t) = E_s \varepsilon_0 + \sum_1^n \varepsilon_0 E_i e^{-t/\tau_i} = E_s \varepsilon_0 + \varepsilon_0 \sum_1^n E_i e^{-t/\tau_i} \tag{g}$$

or,

$$G(t) = \frac{\sigma(t)}{\varepsilon_0} = E_s + \sum_1^n E_i e^{-t/\tau_i} . \tag{g}$$

Here $G(t)$ is the modulus, t the loading time and E_s, E_i, τ_i are parameters. This latter result is commonly known as *prony series* model. It is widely used to characterize relaxation of several materials, especially in numerical simulations. It should be mentioned, however, that in a real experiment, E_i are interpreted as parameters and not as individual moduli and calculated by fitting (g) to the experimental data.

The finite number of elements can be replaced by a continuous function, which reflects a continuous distribution of relaxation times, i.e.

$$G(t) = E_s + \sum_1^n E_i e^{-t/\tau_i} \quad \Rightarrow \; G(t) = E_s + \int_0^\infty E(\tau) e^{-t/\tau_i} d\tau \, . \qquad \text{(f)}$$

3. Kelvin-Voight models in series

We can proceed in a similar manner with the Kelvin-Voight model in series in order to obtain the creep response of a material with several characteristic times and more complicated response. To capture the initial strain response of the material, we add a spring in series with a modulus E_s as shown in Fig. B14. Thus, in creep, each Kelvin-Voight model is subjected to $\sigma_i = \sigma_0 = \sigma$ for $t \geq 0$, and $\varepsilon_i(0) = 0$.

The analysis presented earlier for creep gives for each element,

$$\varepsilon_i(t) = c_i e^{-t/\tau_i} + \frac{\sigma_i}{E_i} \, . \qquad \text{(a)}$$

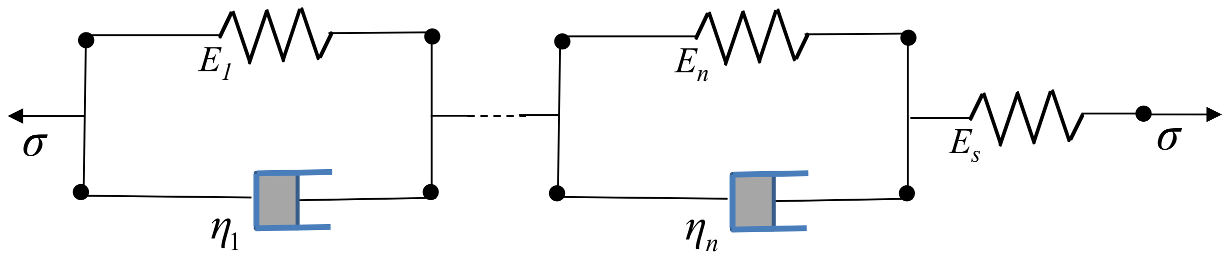

Fig. B14: Kelvin-Voight models and spring in series.

With $\varepsilon_i(0) = 0$ we have,

$$c_i = -\frac{\sigma_i}{E_i} \quad \Rightarrow \quad \varepsilon_i(t) = \frac{\sigma_i}{E_i}\left(1 - e^{-\tau_i/t}\right) . \qquad \text{(b)}$$

The individual spring element is also under constant stress with deformation,

$$\varepsilon_s = \frac{\sigma_s}{E_s} \, . \qquad \text{(c)}$$

The total strain is the sum of each element's strain,

$$\varepsilon(t) = \varepsilon_s + \sum_i^n \varepsilon_i(t) = \varepsilon_s + \sum_i^n \frac{\sigma_i}{E_i}\left(1 - e^{-\tau_i/t}\right) . \qquad \text{(d)}$$

Using the compliance notation and $\sigma_i = \sigma_s = \sigma_0$ we obtain the *creep function*,

$$J(t) = \frac{\varepsilon(t)}{\sigma_0} = J_s(t) + \sum_i^n J_i\left(1 - e^{-\tau_i/t}\right). \quad \text{(e)}$$

For a large number of Kelvin-Voight elements, or when $n \to \infty$, we can model a material with a continuous distribution of retardation times,

$$J(t) = J_s(t) + \int_0^\infty J(\tau)\left(1 - e^{-\tau/t}\right)d\tau. \quad \text{(f)}$$

4. Maxwell and Kelvin-Voight models in series.

A model that combines several characteristics of time depended materials, is shown in Fig. B15. This four-parameter model, is interesting because it can capture the instantaneous elastic response due to the spring E_1, viscous flow due to the dashpot η_1 and delayed response with the Kelvin-Voight model.

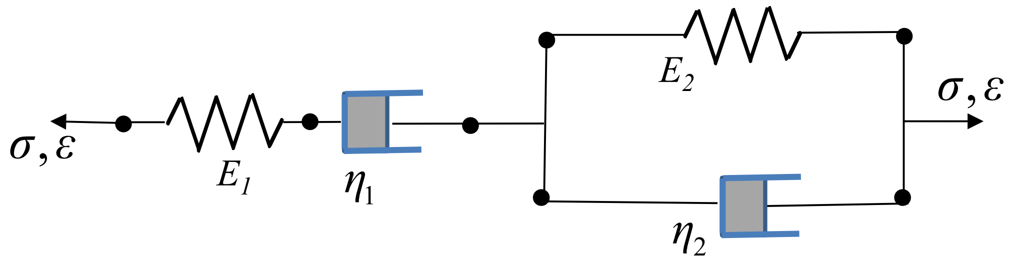

Fig. B15: A four-parameter model.

The constitutive response can be obtained by following procedures similar to the ones in the previous examples and is,

$$\ddot{\sigma}\frac{\eta_1\eta_2}{E_1E_2} + \dot{\sigma}\left(\frac{\eta_1}{E_2} + \frac{\eta_1}{E_1} + \frac{\eta_2}{E_2}\right) + \sigma = \frac{\eta_1\eta_2}{E_2}\ddot{\varepsilon} + \eta_1\dot{\varepsilon} \quad \text{(a)}$$

which is particular case of (B.13). This model is particularly explored in creep and recovery experiments. As an illustration, we will use it below to describe creep. Thus, with $\ddot{\sigma} = \dot{\sigma} = 0$ and $\sigma = \sigma_0$, (a) is simplified to,

$$\ddot{\varepsilon} + \frac{E_2}{\eta_2}\dot{\varepsilon} = \frac{E_2}{\eta_1\eta_2}\sigma_0, \quad \text{(b)}$$

It is not difficult to see that the solution of this 2nd order non-homogeneous differential equation is (sum of the homogeneous and particular solution),

$$\varepsilon(t)=\varepsilon_h(t)+\varepsilon_p(t)=C_1 e^{-\frac{E_2}{\eta_2}t}+C_2+\frac{\sigma_0}{\eta_1}t\,, \tag{c}$$

To proceed further, we need two initial conditions. The first one, is obtained at t = 0. Upon application of the stress, we have the instantaneous response of the spring,

$$\varepsilon(t=0)=C_1+C_2=\frac{\sigma_0}{E_1}\,. \tag{f}$$

A second condition is not so obvious to identify.

To continue, we examine the asymptotes of (c). It is clear that there are no vertical or horizontal ones but an oblique asymptote only. Suppose that this latter is described by,

$$\hat{\varepsilon}(t)=kt+b \tag{g}$$

where,

$$k=\lim_{t\to\infty}\frac{\varepsilon(t)}{t}=\frac{\sigma_0}{\eta_1}=\tan\alpha\ ;\qquad b=\lim_{t\to\infty}\left[\varepsilon(t)-kt\right]=C_2\,. \tag{h}$$

We will next describe creep by combining the creep of each model in series as follows:

1. For the Maxwell model, $\varepsilon_1(t)=\dfrac{\sigma_0}{\eta_1}t+\dfrac{\sigma_0}{E_1}$
2. For the Kelvin-Voight model, $\varepsilon_2(t)=\dfrac{\sigma_0}{E_2}\left(1-e^{-\frac{E_2}{\eta_2}t}\right).$

Accordingly, the total strain of the four-element model under creep is,

$$\begin{aligned}\varepsilon(t)&=\varepsilon_1(t)+\varepsilon_2(t)=\frac{\sigma_0}{E_2}\left(1-e^{-\frac{E_2}{\eta_2}t}\right)+\frac{\sigma_0}{\eta_1}t+\frac{\sigma_0}{E_1}=\frac{\sigma_0}{E_2}-\frac{\sigma_0}{E_2}e^{-\frac{E_2}{\eta_2}t}+\frac{\sigma_0}{\eta_1}t+\frac{\sigma_0}{E_1}\\ \varepsilon(t)&=-\frac{\sigma_0}{E_2}e^{-\frac{E_2}{\eta_2}t}+\frac{\sigma_0}{E_2}+\frac{\sigma_0}{E_1}+\frac{\sigma_0}{\eta_1}t\end{aligned}\,. \tag{i}$$

The last equation has the asymptote given by (g) and thus, $b = \frac{\sigma_0}{E_1} + \frac{\sigma_0}{E_2}$. It is also evident that the creep rate at the start of the experiment, given by the first derivative of the solution (i), is,

$$\left.\frac{d\varepsilon}{dt}\right|_{t=0} = \frac{\sigma_0}{\eta_2} + \frac{\sigma_0}{\eta_1} = \tan\beta \,. \qquad \text{(g)}$$

The creep response of the model, the asymptote and the initial strain rate are schematically shown in Fig. B16. Interestingly, the model in Fig. B16, gives a good trend of creep in Fig. B6 and could fit the experimental data. However, a complete analysis would require the identification of the model's parameters fitting the experimental data in equation (i).

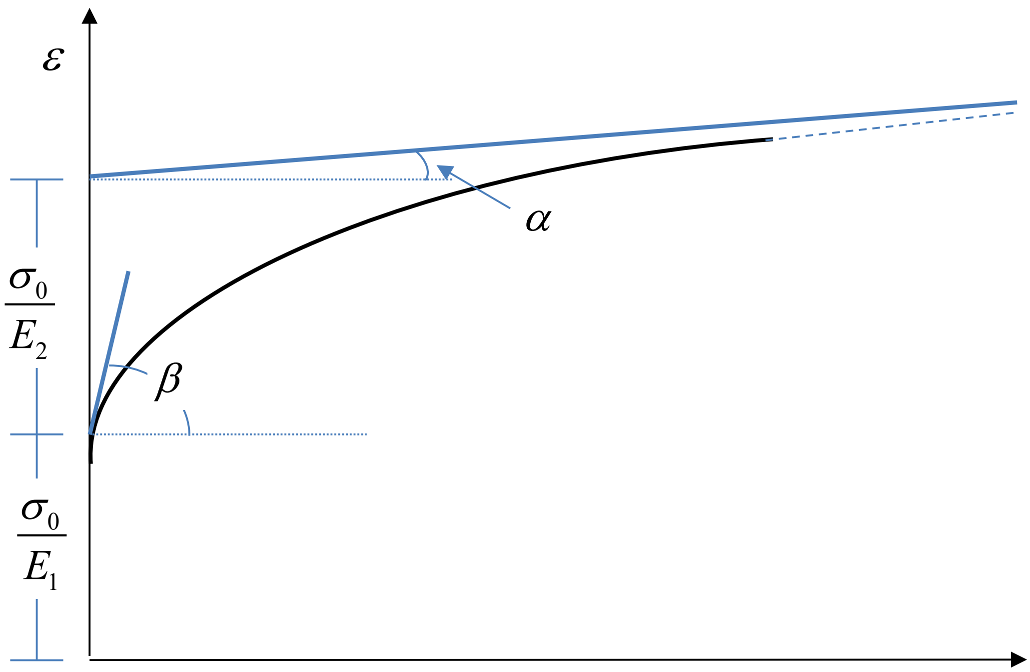

Fig. B16: Creep response of the four-parameter model in Fig. B16.

Summary

It should be noted that viscoelastic characterization of real materials, (rubbers, polymers, composites, gels, asphalts, …), is based on models summarized in this short review combined with additional elements to take into consideration certain specific behaviors. In the literature, the differential equations for each combination of models are called constitutive, equations of state, or rheological equations of state.

The material parameters (moduli and viscosity coefficients) are calculated by fitting each model's equation to experimental data. The resulting equations are often used in numerical simulations to predict material or structural response under creep or relaxation. Cyclic loading is also very important is viscoelastic solids. However, it outside the scope of this short review.

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